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A calculus in differentiable spaces and its application to loops

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Abstract. Recently, a set of tools has been developed with the purpose of studying quantum gravity. Until now, there have been very few attempts to put these tools into a rigorous mathematical framework. This is the case, for example, for the so-called *path bundle* of a manifold. It is well known that this topological principal bundle plays the role of a universal bundle for the reconstruction of principal bundles and their connections. The path bundle is canonically endowed with parallel transport and, associated with it, important types of derivatives have been considered by several authors: the Mandelstam derivative, the connection derivative and the loop derivative. Here we shall give a unified viewpoint for all of these derivatives by developing a differentiable calculus on differentiable spaces. In particular, we shall show that the loop derivative is the curvature of a canonically defined 1-form that we shall call the *universal connection 1-form*.

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1. Introduction

The use of paths for the study of geometric properties of manifolds have been shown to be a powerful and rich tool. This seems to be due to the fact that paths naturally carry information between their two extreme points. One example of such a transit of information is given by parallel transport, and its use introduces a sort of nonlinear duality between loops (a particular equivalence class of piecewise paths) and connections. This duality is the root for the loop representation of Yang–Mills quantum theories. It began in the work of Mandelstam [10, 11] and was followed by many others. Since the discovery by Ashtekar of a new set of variables, making gravity closer to a gauge Yang–Mills theory than geometrodynamics, much of the attention on canonical quantization for gravity has turned to trying to find a loop representation of it. In particular, it has led to solving (2 + 1)-dimensional gravity exactly [2]. After the importance of loops in gauge theory was really understood many people tried to realize a rigorous theory for it and their calculus. There are several efforts in this respect that cover different topics of the problem of the reconstruction of connections by their holonomies or certain functions of it [3], definitions of derivatives [8] and the problem of a suitable definition of the group of a loop for each particular gauge group [1].

In this paper we try a unified viewpoint for all of these results, developing a differential calculus on differentiable spaces first and then applying it to the group of loops and the path bundle. The use of differentiable spaces is inspired by the work of Chen [5], although we follow a different direction in the definition of vector fields and differentials forms. Most of the work is also based on the papers by Barrett [3] and Lewandowski [9].

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The paper is organized as follows, in section 2 we recall the definition of the group of loops and the path bundle.

In section 3 we define *differentiable spaces* as a set A with a family of plots (functions from open sets of $\mathbb{R}^n$ to A), then we can say that a function from a differentiable space A to $\mathbb{R}$ is differentiable, it is when its composition with every plot is differentiable. As examples of differentiable spaces we can mention the following.

- The group of loops $\mathcal{L}$, with plots defined as functions from U to $\mathcal{L}$ induced by piecewise differentiable functions $f\colon U \times [0, 1] \to M$ such that $f_x(0) = f_x(1) = o$. The same is true for the path bundle $\mathcal{PM}$ but with free ends.
- Differentiable manifolds with plots taken to be all differentiable functions from open subsets of $\mathbb{R}^n$ to the manifold.
- The group of diffeomorphisms of M, $\mathrm{Diff}(M)$, with plots taken as functions from U to $\mathrm{Diff}(M)$, induced by differentiable functions from $U \times M \to M$.
- Spin networks with plots generated from functions $\Phi\colon U \times \Gamma \to M$ where Γ is an abstract graph, that is Φ is a family of embeddings of Γ in M parametrized by U.

Having this notion of differentiable space we define differentiable functions as mentioned above and define a vector tangent to a point $x \in A$ as differential operators (see definition 3.2) $D\colon C^\infty(A) \to \mathbb{R}$ satisfying the Leibnitz rule $D(fg) = D(f)\, g(x) + f(x)\, D(g)$ and vector fields on A as differential operators $X\colon C^\infty(A) \to C^\infty(A)$ satisfying $X(fg) = X(f)\, g + f\, X(g)$. Differential forms are defined as usually as multilinear alternated functions of tangent vectors and exterior derivatives are defined by a familiar formula of differential geometry (see definition 3.6). These constructions have the usual properties of the corresponding objects in finite-dimensional differentiable manifolds, in fact, they are the same when a differentiable manifold is considered as a differentiable space, and other constructions such as fibre bundles and connections can be generalized to differentiable spaces in a straightforward way. We remark that the use of differentiable spaces is valuable in itself and most of the work in this paper is devoted to showing the generality and usefulness of this kind of spaces.

In section 4 we shall recall the definition of the Mandelstam derivative. In section 5 we shall define the *universal connection 1-form*. This will be a differential 1-form in the sense of section 3 and we shall be able to express the connection derivative in terms of this universal connection. Section 6 contains our main result. We shall prove that the loop derivative considered in [8] is nothing but the curvature of the universal connection 1-form. Consequently, we obtain the Bianchi identities in [8] as the usual Bianchi identities associated with the universal connection 1-form. Finally, in section 7 we shall see how to represent a particular gauge theory using the results we obtained in the previous sections.

2. Group of loops, path bundle and parallel transport

The group of loops of a manifold M is defined in the following way. Let o be a fixed point in M, and let L be the set of piecewise smooth paths $\alpha(t)$, parametrized from [0, 1] such that $\alpha(0) = \alpha(1) = o$ and PM be the set of paths such that $\alpha(0) = o$. In the space of paths we define the product of two paths α and β such that $\alpha(1) = \beta(0)$ by $\alpha \cdot \beta(t)$ is $\alpha(2t)$ if $t < \frac{1}{2}$ or $\beta(2t-1)$ if $t \geqslant \frac{1}{2}$. And define the inverse path α^{-1} by $\alpha^{-1}(t) = \alpha(1-t)$. In L we consider the equivalence relation that identifies paths that differ by an orientation-preserving reparametrization, then we say that two paths α and β are elementary equivalent if there exists ρ, ξ, γ such that $\alpha = \rho \cdot \xi$ and $\beta = \rho \cdot \gamma \cdot \gamma^{-1} \cdot \xi$ and we define the equivalence relation $\alpha \approx \beta$ iff there exist a sequence of paths $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta$ with α_i elementary equivalent to α_{i+1}.

Definition 2.1. *The group of loops is the quotient space of L by this equivalence relation and is denoted by $\mathcal{L}$.*

Observation 2.2. *$\mathcal{L}$ with the product of paths has a group structure with inverse α^{-1} and identity the constant path.*

There exist others possibilities for the definition of the equivalence relation that can lead to different loop groups [8, 1, 13].

Let PM denote the set of piecewise smooth paths $\alpha : [0,1] \to M$, with $\alpha(0) = o$.

Definition 2.3. *The path bundle, denoted by $\mathcal{PM}$, is the quotient space of PM by the same equivalence relation that for $\mathcal{L}$.*

The group of loops acts on $\mathcal{PM}$ by left multiplication. As is explained in next section there exists a topology, the so-called Barrett topology, that makes $(\mathcal{PM}, M, \mathcal{L}, \Pi)$ the topological principal $\mathcal{L}$-bundle, where $\Pi\colon \mathcal{PM} \to M$ is the function that assigns to each path its endpoint, thus the fibre over x, $\Pi^{-1}(x)$ is the set of paths from o to x module the above-mentioned equivalence relation.

There is a canonical way to define the parallel transport in this bundle, given a path γ in M with initial point x and final point y, and an element $[\alpha]$ of the fibre of $\mathcal{PM}$ over x, where α is a path going from o to x, the parallel transport of $[\alpha]$ over γ is $[\alpha \cdot \gamma]$, which is an element of the fibre over y.

3. Differentiable structure

In this section we define differentiable functions, tangent vectors, vector fields and differential forms for the group of loops and the path bundle.

To define differentiable functions we follow the idea of Barrett [3] and define a *homotopy* of paths to be a function from an open set U of $\mathbb{R}^n$ to $\mathcal{PM}$ (or $\mathcal{L}$), $\Phi\colon U \to \mathcal{PM}$ that is obtained from a function $\phi\colon U \times [0,1] \to M$, such that there exists a partition of $[0,1]$, $0 = i_0 < i_1 < \cdots < i_n = 1$, such that ϕ is differentiable in $U \times [i_k, i_{k+1}]$. Then we say that a function $f\colon \mathcal{PM} \to \mathbb{R}$ is differentiable if its composition with every homotopy is differentiable.

We will advance further and define concepts such as vector fields and differential forms, but these concepts seems to appear more naturally if we work in a more general framework.

3.1. Differentiable spaces

Definition 3.1. *A* differentiable space *is a set A with a family of functions from open subsets of $\mathbb{R}^n$ to A, called* plots, *such that if $\Phi\colon U \to A$ is a plot and if $g\colon V \to U$ is differentiable, $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^k$ then $\Phi \circ g\colon V \to A$ is a plot.*

We endow a differentiable space A with the topology induced by the plots, that is, a subset $U \subset A$ is open if and only if $\Phi^{-1}(U)$ is open for every plot Φ.

The group of loops and the path bundle are differentiable spaces with the plots taken to be homotopies, and the topology considered is the Barrett topology.

As other examples of differentiable spaces we could mention a differentiable manifold considering all the differentiable functions from open subsets of $\mathbb{R}^n$ to M as plots, and the diffeomorphism group of M, $\mathrm{Diff}(M)$, considering as plots the functions from open sets U of $\mathbb{R}^n$ to $\mathrm{Diff}(M)$ induced by differentiable functions from $U \times M \to M$.

The other way of constructing new differentiable spaces is by products of other differentiable spaces. Given two differentiable spaces A and B we can give to their product

$A \times B$ a structure of differentiable space; the plots of $A \times B$ are constructed as (Φ, Φ') where Φ is a plot of A and Φ' is a plot of B and any composition of these with differentiable functions.

3.2. Differentiable functions

Generalizing the definition given for the path bundle and the group of loops we define a function $f: A \to \mathbb{R}$ from a differentiable space A to be differentiable if and only if its composition with every plot of A is differentiable. For functions $f: A \to B$ between two differentiable spaces, we consider f differentiable iff the composition of f with every plot of A is a plot of B. For example, the product of loops $\cdot: \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ and the action of $\mathcal{L}$ over $\mathcal{PM}$, $\cdot: \mathcal{L} \times \mathcal{PM} \to \mathcal{PM}$ are differentiable.

3.3. Tangent vectors

We want to define tangent vectors in the group of loops, remembering the definition of vectors in a manifold as directional derivatives, it is natural to define tangent vectors in the group of loops, as Lewandowski does [9], as operators D acting in the space $C^\infty(\mathcal{L})$ such that

$$Df = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} f \circ \Phi$$

where Φ is a homotopy $\Phi: (-\epsilon, \epsilon) \to \mathcal{L}$. However, in our context that definition is not appropriate, the loop derivative is defined by means of a second-order derivative (see definition 6.1) and it is not clear how to define the bracket of vector fields. We are thus led to consider higher-order derivatives.

We will use the multi-index notation. A multi-index is an n-tuple of non-negative integers $\alpha = (a_1, a_2, \ldots, a_n)$, $|\alpha| = a_1 + \cdots + a_n$ and

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{a_1} \ldots \partial x_n^{a_n}},$$

where all the derivatives are taken in $x_i = 0$.

Definition 3.2. *Let x_0 be a point of A, an* elemental differential operator at x_0 *is a linear transformation $D: C^\infty(A) \to \mathbb{R}$ such that there exists a plot $\Phi: U \to A$, $\Phi(0) = x_0$ and a multi-index α such that*

$$D(f) = \partial^\alpha(f \circ \Phi), \qquad \forall f \in C^\infty(A).$$

The space of differential operators at x_0, *denoted by $D_{x_0}A$ is the vector space generated by the elemental differential operators.*

Thus a differential operator at x_0 is a finite linear combination of elemental differential operators at x_0.

We now define the order of a differential operator. Let m_{x_0} be the ideal in $C^\infty(A)$ of the functions that vanish in x_0, and $m^n_{x_0}$ be the set of linear combination of products of n functions that vanish in x_0. We define the *order* of a differential operator at x_0, D, as the minimum n such that $D|m^{n+1}_{x_0} = 0$. It is straightforward to check that a differential operator at x_0, D, is of first order $D(fg) = D(f)\, g(x_0) + f(x_0) D(g)$, this motivates the following definition.

Definition 3.3. *The* tangent vectors *of a space A at the point x_0 are the first-order differential operators at x_0 and the vector space of all tangent vectors at x_0 is the tangent space of A at x_0, denoted as $T_{x_0}A$.*

When we have a function $f: A \times B \to \mathbb{R}$ and $D \in T_{x_0}A$, $D' \in T_{y_0}B$ then $D_x f(x, y)$, meaning the differential operator applied only on the first variable, is a differentiable function from B to $\mathbb{R}$, and $D_x D'_y f(x, y) = D'_y D_x f(x, y)$ the differential operators applied in different variables commutes, the proof is immediate. When we have $f: A \times A \to \mathbb{R}$, $D' \in T_{y_0}B$ then

$$D_x f(x, x) = D_x f(x, x_0) + D_x f(x_0, x),$$

the proof of this requires the following theorem.

Theorem 3.4. $T_{(x_0,y_0)}A \times B = T_{x_0}A \oplus T_{y_0}B$.

Proof. If D is an elemental differential operator (diff. op.) at (x_0, y_0) we can see from the definition that D can be written as

$$D = \bar{D}_0 + \hat{D}_0 + \sum_{i=1}^{n} \bar{D}_i \hat{D}_i,$$

where $\bar{D}$ are diff. op. at x_0 and $\hat{D}$ are diff. op. at y_0, then any differential operator can be written in this way.

Next we impose that the operator D be of first order, considering the cases $g(x, y) = f(x)$ and $g(x, y) = h(y)$ we see that $\bar{D}_0$ and $\hat{D}_0$ are of first order, then $D' = \sum_{i=1}^{n} \bar{D}_i \hat{D}_i$ is also of first order.

We want to prove that D' is null, by absurdity, let us assume that it is not null.

The operator D' can be written in many ways as a sum $\sum_{i=1}^{n} \bar{D}_i \hat{D}_i$, we choose those with less number of terms, then $\bar{D}_i$ are linearly independent (if not we can combine terms to obtain a sum with fewer terms).

$\hat{D}_1$ is not null (if not we can eliminate the term $\bar{D}_1 \hat{D}_1$ of the sum) then there is a function g that vanishes in y_0 such that $\hat{D}_1(g) \neq 0$; for all functions f that vanish in x_0 we define $h(x, y) = f(x)\, g(y)$ then $D'(h) = 0$ because h is a product of two functions that vanish in (x_0, y_0) and D' is of first order, then

$$\bar{D}_1(f)\hat{D}_1(g) + \cdots + \bar{D}_n(f)\hat{D}_n(g) = 0.$$

for all f and $\hat{D}_1(g) \neq 0$, then $\bar{D}_i$ are linearly dependent, which is absurd. □

Given a differentiable function $f: A \to B$ where A and B are differentiable spaces, then we can define its differential as a map from $T_x A$ to $T_{f(x)}B$, let $D \in T_x A$, the operator $d_x f(D)$ is defined as $d_x f(D)(g) = D(g \circ f)$, where g is any function on $C^\infty(B)$.

3.4. Tangent bundle

We can define the tangent bundle TA of a differentiable space A as the disjoint union of the tangent spaces, and it can be given a differentiable structure, for this we take as an auxiliary construction the differential bundle DA defined as the disjoint union of the spaces of differential operators at x for every $x \in A$, and the projection $\pi: DA \to A$ such that $\pi(D) = x$ iff D is a differential operator in x.

Given a plot $\Phi: V \times U \to A$ we define an elemental plot $\Psi: U \to DA$ as $\Psi = \partial^\alpha \Phi$ where the derivation is taken only with respect to the first variables (those which lie in V), that is $\Psi(x)$ is the differential operator in $\Phi(0, x)$ such that $\Psi(x)(f) = \partial_y^\alpha f(\Phi(y, x))$. Then we define the plots in DA as finite linear combinations of elemental plots $\Psi = a_1\Psi_1 + \cdots + a_n\Psi_n$ where $\Psi_i: U \to DA$ are elemental plots and $\pi \circ \Psi_i = \pi \circ \Psi_j$, in this way $\Psi_i(x)$ are differential operators over the same point, the sum is a differential operator at that point. $TA \subset DA$ thus we define a plot in TA as a function $\Phi: U \to TA$ that is a plot in DA.

We now investigate when the tangent bundle is locally trivial, that is for every point $x \in A$ there is a neighbourhood U of x such that TU is diffeomorphic to $U \times T_x A$ with a diffeomorphism that is linear restricted to each tangent space. A sufficient condition for this is the existence for every point $x \in A$ of a neighbourhood U of x and a differentiable function $f: U \times U \times U \to A$. We can see this f as a function $f_{x,y} : U \to A$ indexed by $x, y \in U$, such that $f_{x,y}(y) = x$, and $f_{x,y}$ is a local diffeomorphism, then the differential of $f_{x,y}$ is a function from $T_y A$ to $T_x A$, thus we can construct a diffeomorphism $\phi: TU \to U \times T_x A$ as

$$\phi(D) = d_{\pi(D)} f_{x,\pi(D)}(D).$$

In the case of $\mathcal{L}$ we construct the function f simply as $f(\alpha, \beta, \gamma) = \alpha \cdot \beta^{-1} \cdot \gamma$. In $\mathcal{PM}$ we define $f(\alpha, \beta, \gamma): \Pi^{-1}(U) \times \Pi^{-1}(U) \times \Pi^{-1}(U) \to \mathcal{PM}$ where U is a convex neighbourhood of the final point of γ as $f(\alpha, \beta, \gamma) = \alpha \cdot \epsilon_{\alpha,\beta} \cdot \beta^{-1} \cdot \gamma \cdot \epsilon$ where $\epsilon_{\alpha,\beta}$ is the straight line joining the end points of α and β, and ϵ is $\epsilon_{\alpha,\beta}$ moved to the final point of γ as we can see in figure 1.

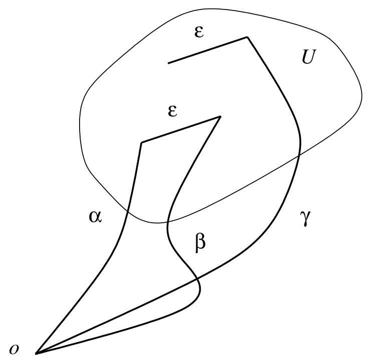

Figure 1. Plot of $f(\alpha, \beta, \gamma)$.

3.5. *Vector fields*

We define vector fields as sections of the tangent bundle, a section is a differentiable function $X: A \to TA$, such that $\pi \circ X = \mathrm{i}d$, that is $X(x)$ is a tangent vector over the point x. Given a section X there is an associated operator $\hat{X}: C^\infty(A) \to C^\infty(A)$ such that $\hat{X}(f)(x) = X(x)f$. Similarly, we can define sections in DA and its associated operators, then we have:

- A section of DA is a section of TA (vector field) iff it satisfies the Leibnitz rule.
- If D_1, D_2 are sections of DA then $D_1 D_2$ (considered as a operator) is a section of DA.
- As a consequence of the latter, the bracket of two vector fields is well defined.

The fact that the bracket is well defined permits us to define the Lie algebra of $\mathcal{L}$ as the space of right-invariant vector fields in $\mathcal{L}$. A vector field $X: \mathcal{L} \to T\mathcal{L}$ is right invariant if for every $\gamma \in \mathcal{L}$, $\mathrm{d}R_\gamma \circ X = X \circ R_\gamma$, where $R_\gamma: \mathcal{L} \to \mathcal{L}$ is the multiplication by γ on the right $R_\gamma(\alpha) = \alpha \cdot \gamma$ and $\mathrm{d}R_\gamma: T\mathcal{L} \to T\mathcal{L}$ is the differential of R_g. The Lie bracket of two elements of the Lie algebra of $\mathcal{L}$ is simply the bracket of the right-invariant vector field.

3.6. *Differential forms*

The differential forms are defined as usual in differential geometry. First we define $\bigoplus^p TA$ as the fibre bundle over A such that the fibre over each element x is $\bigoplus^p T_x A$, that is a element of $\bigoplus^p TA$ is a p-tuple $(v_1, \ldots, v_n)$ where $v_1, \ldots, v_i$ are tangent vectors to A over the same point.

Definition 3.5. *A p-form ω is defined as an alternated, multilinear differentiable function $\omega\colon \bigoplus^p TA \to \mathbb{R}$.*

This means that for every x $\omega_x\colon (T_xA)^p \to \mathbb{R}$ is a multilinear and alternated function. The differentiability of ω implies that for every p-tuple of vector fields $X_1, \dots, X_n$ the function $x \mapsto \omega_x(X_1(x), \dots, X_p(x))$, denoted simply by $\omega(X_1, \dots, X_p)$, is differentiable.

Next we define the exterior derivative of a form, now we require that the tangent bundle of A be locally trivial. Because of local triviality of the tangent bundle every vector $v \in T_xA$ can be extended locally to a vector field $X\colon U \to TA$, such that U is a neighbourhood of x and $X(x) = v$, thus it is correct to define the exterior derivative $d\omega_x(v_1, \dots, v_n)$ using vector fields $X_1, \dots, X_n$ such that $X_i(x) = v_i$ if we prove that the result only depends on $v_1, \dots, v_n$.

Definition 3.6. *The* exterior derivative *of ω is defined as*

$$\begin{aligned} d\omega(X_1, \dots, X_{n+1}) = \frac{1}{n+1}\Bigg[&\sum_{i=1}^{n+1} (-1)^{i+1} X_i(\omega(X_1, \dots, \hat{X}_i, \dots, X_{n+1})) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{n+1})\Bigg]. \end{aligned}$$

By the formula we see that $d\omega$ is also multilinear and alternated, but it remain to be proved that it is local too. Let us prove this in the next lemma.

Lemma 3.7. $d\omega(X_1, \dots, X_n)(x)$ *only depends on the values $X_i(x)$.*

Proof. Because of linearity, it is sufficient to show that $d\omega(X_1, \dots, X_{n+1})(x) = 0$ when $X_1(x) = 0$. Looking at the formula it is also sufficient to show that

$$X_i(x)\,(\omega(X_1, \dots, \hat{X}_i, \dots, X_{n+1})) + \omega([X_1, X_i], X_2, \dots, \hat{X}_i, \dots, X_{n+1})\,(x) = 0.$$

Let us consider the 1-form $\omega(X) = \omega(X, X_2, \dots, \hat{X}_i, \dots, X_{n+1})$ then what we have to prove is

$$Y(x)\,\omega(X) + \omega([X, Y])\,(x) = 0$$

whenever $X(x) = 0$.

Next we construct, using a local trivialization of the tangent bundle, a function $X\colon U \times U \to TA$, where U is a neighbourhood of x that verifies

- $X(y, z) \in T_yA$
- $X(y, y) = X(y)$
- $X(y, x) = 0$.

The proof then follows looking at the equalities

$$\begin{aligned} Y_y(x)\,\omega(X(y, y)) &= Y_y(x)\,\omega(X(y, x)) + Y_y(x)\,\omega(X(x, y)) \\ &= Y_y(x)\,\omega(X(x, y)) = \omega(Y_y(x)\,X(x, y)) \\ [X, Y](x) &= -Y_y(x)\,(X(y)) = -Y_y(x)\,(X(y, y)) \\ &= -Y_y(x)\,(X(y, x)) - Y_y(x)\,(X(x, y)) = -Y_y(x)\,(X(x, y)) \end{aligned}$$

□

4. Mandelstam derivative

The parallel transport defined in section 2 induces a set of planes in $\mathcal{PM}$ that will be the horizontal planes of the universal connection 1-form that we shall define later.

Definition 4.1. *The Mandelstam derivative on π_o^x in the direction of $v \in T_x M$ is given by*

$$D_v f(\pi_o^x) = \mathrm{d}_x(f \circ \Phi)(v)$$

where ϕ is a family of curves represented by

$$\phi(y,t) = \begin{cases} \pi_o^x(2t) & 0 \leqslant t \leqslant \frac{1}{2} \\ g((2t-1)\, g^{-1}(y)) & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

and

$$\Phi(y) = [t \mapsto \phi(y,t)]$$

where $g: U \to M$ is a chart with U a convex neighbourhood of 0 and $g(0) = x$.

Observation 4.2. *If we call δv the segment (in the chart g) from x to $x + \epsilon v$ then*

$$f(\pi_o^x \cdot \delta v) = f(\pi_o^x) + \epsilon D_v f + o(\epsilon).$$

Thus the Mandelstam derivative coincides with the one defined in [8].

It needs to be proved that the definition does not depend of the chart g, for that we first state a lemma proved in [3].

Lemma 4.3. *If $\phi: (-\epsilon, \epsilon) \times [0,1] \to M$ is a homotopy of loops where $\phi(0,t) = o$ is the constant loop and $\Phi(s) = [t \mapsto \phi(s,t)]$ then for any differentiable function $f: \mathcal{L} \to \mathbb{R}$,*

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} f \circ \Phi = 0.$$

Proof. $\phi(0,t) = o$ for all t in $[0,1]$ so we can assume that $\phi(s,t)$ is contained in a neighbourhood of o for all $s \in (-\epsilon, \epsilon)$ and we can choose that neighbourhood such that there is a local chart of M, $g: U \to M$, $g(0) = o$. We consider $\phi: (-\epsilon, \epsilon) \times [0,1] \to \mathbb{R}^n$, then $\phi(s,t) = (\phi_1(s,t), \ldots, \phi_n(s,t))$, let

$$\phi'(s_1, \ldots, s_n, t) = (\phi_1(s_1,t), \ldots, \phi_n(s_n,t)),$$

and Φ' the associated function from $(-\epsilon, \epsilon)^n$ to $\mathcal{PM}$, let $\Delta: \mathbb{R} \to \mathbb{R}^n$ be the diagonal function, then $\phi = \phi' \circ \Delta$, and $\Phi = \Phi' \circ \Delta$, then

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} f \circ \Phi = \frac{\partial}{\partial s_1} f \circ \Phi' + \cdots + \frac{\partial}{\partial s_n} f \circ \Phi'$$

but when all the s_i except one are zero the loop Φ' is contained in a segment, and hence is null, then $\partial/\partial s_i\, f \circ \Phi' = 0$. □

Proposition 4.4. *Let $\alpha: (-\epsilon, \epsilon) \to M$ be any curve with $\alpha(0) = x$ and $\dot{\alpha}(0) = v$, define the homotopy of paths $\Phi(s) = \pi_o^x \cdot [t \mapsto \alpha(st)]$, and let the operator $\tilde{D}_v$ be such that*

$$\tilde{D}_v(f) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} f \circ \Phi$$

then $\tilde{D}_v = D_v$.

Proof. Let $\alpha(s) = (\alpha_1(s), \ldots, \alpha_n(s))$ in the chart g and assume that in that chart $v = e_1$ where $e_1, \ldots, e_n$ is the canonical basis of $\mathbb{R}^n$, then $\alpha'_1 = 1$ and $\alpha'_i = 0$, $i = 2, \ldots, n$.

We construct the homotopy of loops Φ such that $\Phi(s)$ is the loop based at x formed by the composition of the curve α from x to $\alpha(s)$ then the straight segment (in the chart g) from $\alpha(s)$ to $g((\alpha_1(s), 0, \ldots, 0))$ and then the segment to x.

By lemma 4.3 $(\mathrm{d}/\mathrm{d}s)|_{s=0} f(\pi_o^x \cdot \Phi(s)) = 0$ and expanding $f(\pi_o^x \cdot \Phi(s))$ until first order in s we obtain

$$f(\pi_o^x \cdot \Phi(s)) = f(\pi_o^x) + s\tilde{D}_v(f) + \sum_{i=2}^{n} \alpha_i(s) D_{e_i}(f) - \alpha_1(s)\, D_{e_1}(f),$$

taking limit when $s \to 0$ we obtain $\tilde{D}_v(f) = D_v(f)$. □

5. Universal connection 1-form

Following which is the habitual definition of the connection 1-form of the theory of principal bundles, we define a connection in $\mathcal{PM}$, that is a 1-form evaluated in the Lie algebra $= T_e\mathcal{L}$ of $\mathcal{L}$, such that it is equivariant under the action of the group and is the identity over the vertical subspaces [6].

Given $D \in T_\pi\mathcal{PM}$ and $f \in C^\infty(\mathcal{L})$ we first transform f into a function on an open subset of $\mathcal{PM}$ given by $g(\gamma) = f(\gamma \cdot v(\gamma) \cdot \pi^{-1})$, where $v(\gamma)$ is the segment that joins $\Pi(\gamma)$ (the final point of γ) with $\Pi(\pi)$ (the final point of π) in a chart $h: U \to M$ with U a convex neighbourhood of 0 and $h(0) = \Pi(\pi)$. We define $\delta_\pi(D)(f) = D(g)$.

Lemma 5.1. *δ is a 1-form in $\mathcal{PM}$ with values in $T_e\mathcal{L}$.*

Proof. First we must show that the definition does not depend of the chart h chosen, for that we investigate how a vector $D \in T_\pi\mathcal{PM}$ can be decomposed into a horizontal and a vertical part:

$$\begin{aligned} D_\gamma f(\gamma) &= D_\gamma f(\gamma \cdot v(\gamma) \cdot \pi^{-1} \cdot \pi \cdot v(\gamma)^{-1}) \\ &= D_\gamma f(\gamma \cdot v(\gamma) \cdot \pi^{-1}) + D_\gamma f(\pi \cdot v(\gamma)^{-1}) = \mathrm{d}U(\delta_\pi(D))\,(f) + D_v(f) \end{aligned}$$

where $v = \mathrm{d}\Pi(D)$ and $U: \mathcal{L} \to \mathcal{PM}$, $U(\xi) = \xi \cdot \pi$; because D_v does not depend of the chart we see that δ_π does not depend on the chart.

To verify that δ is a 1-form in $\mathcal{PM}$ evaluated in $T_e\mathcal{L}$ we define the function $g: \Pi^{-1}(U) \times \Pi^{-1}(U) \to \mathcal{L}$, $g(\gamma, \pi) = \gamma \cdot v \cdot \pi^{-1}$, then $\delta_\pi(D)(f) = d_{\gamma=\pi} g(\gamma, \pi)(D)$, thus we can see that δ is differentiable. □

Let us see that δ is really a connection.

Lemma 5.2. *δ satisfies the definition of a connection.*

Proof. To prove this we have to see first that it is the identity over the vertical vectors and must verify the compatibility condition over the action of the group.

Let U be $U: \mathcal{L} \to \mathcal{PM}$, $U(\gamma) = \gamma \cdot \pi$, the vertical vectors in $T_\pi PM$ are those that lie at the image of $\mathrm{d}U$, and making the calculation $\delta_\pi \circ \mathrm{d}U$ we obtain

$$\delta_\pi(\mathrm{d}U(D))\,(f) = \mathrm{d}U(D)_\gamma f(\gamma \cdot v \cdot \pi^{-1}) = D_{\gamma'} f(\gamma' \cdot \pi \cdot v \cdot \pi^{-1}) = D_{\gamma'} f(\gamma') = D(f).$$

Let U_α be $U_\alpha: \mathcal{PM} \to \mathcal{PM}$, $U_\alpha(\pi) = \alpha \cdot \pi$,

$$\begin{aligned} (U_\alpha^*\delta)_\pi(D)\,(f) &= \delta_{\alpha\cdot\pi}(\mathrm{d}U_\alpha(D))\,(f) = \mathrm{d}U_\alpha(D)_\gamma f(\gamma \cdot v \cdot \pi^{-1} \cdot \alpha^{-1}) \\ &= D_{\pi'} f(\alpha \cdot \pi' \cdot v \cdot \pi^{-1} \cdot \alpha^{-1}) = Ad(\alpha)\delta_\pi(D)(f). \end{aligned}$$ □

As an illustrative example let us see how the connection derivative given in [8] enters into this context. Let us take a section of the bundle around the point of the fibre over x, π_o^x. That means an election of a family of curves $\Phi\colon U \to \mathcal{PM}$ where U is an open neighbourhood of the x point $y\, \Pi(\Phi(y)) = y$, $y \in U$. Given a function $f(\pi_o^x)$ we define the tangent vector to the section Φ as the operator that applied to f gives

$$\tilde{D}_v f(\pi_0^x) = \partial_v f \circ \Phi(x) = \frac{\mathrm{d} f(\pi_0^{x+\epsilon v})}{\mathrm{d}\epsilon}.$$

We observe that the projection of this vector over M is v.

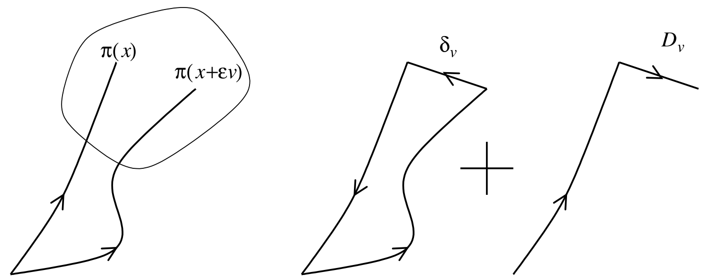

Figure 2. The figure on the left represent a trivialization. Those on the right are the vertical and horizontal vectors decomposing the tangent vector into the trivialization.

6. The loop derivative as a curvature

In this section, we will see that the loop derivative defined in [8] represents the curvature of the universal connection 1-form defined above.

Definition 6.1. *The loop derivative is a vector field in $\mathcal{L}$ that applied to a function $f(\gamma)$, $\gamma \in \mathcal{L}$ is*

$$\Delta_{u,v}(\pi_o^x) f(\gamma) = \frac{\partial^2 f(\pi \cdot \Box \cdot \pi^{-1} \cdot \gamma)}{\partial\epsilon_1 \partial\epsilon_2}$$

where $\Box = \Box_{\epsilon_1,\epsilon_2}$ is the parallelogram, taken in a local chart of M, with vertex at x and edges in the directions of the vectors u and v with lengths $\epsilon_1 \cdot \|u\|$ and $\epsilon_2 \cdot \|v\|$, respectively.

Its needs to be checked that the loop derivative does not depend on the chart used to define the parallelogram $\Box$, we postpone this until the end of this section.

The curvature form is written in terms of the connection form as (cf [6])

$$\Omega = \mathrm{d}\delta + \tfrac{1}{2}[\delta, \delta].$$

Since the curvature form is horizontal it suffices to evaluate it on horizontal vectors, that is Mandelstam derivatives D_μ, D_ν that are horizontal vector fields defined in a neighbourhood of the point x.

In this context applying the expression for the exterior derivative given in section 3, and evaluating the curvature on the horizontal fields D_μ, D_ν we obtain

$$\begin{aligned}\Omega(D_\mu, D_\nu) &= \tfrac{1}{2}\big[D_\mu\delta(D_\nu) - D_\nu\delta(D_\mu) - \delta([D_\mu, D_\nu]) + [\delta(D_\mu), \delta(D_\nu)]\big]\\ &= -\tfrac{1}{2}\delta([D_\mu, D_\nu]) = -\tfrac{1}{2}[D_\mu, D_\nu],\end{aligned}$$

because the connection form is null on horizontal vectors and is the identity on vertical vectors and $[D_\mu, D_\nu]$ is vertical.

In [8] it is proved that

$$[D_\mu, D_\nu] = \Delta_{\mu,\nu},$$

thus

$$\Delta_{\mu,\nu} = -2\Omega(D_\mu, D_\nu).$$

This also proves that the loop derivative does not depend on the chart taken in the definition, because the Mandelstam derivative does not depend on any chart.

In this context the Bianchi identity presented in [8] follows naturally from the Bianchi identity for the curvature in $\mathcal{PM}$

$$\mathrm{d}\Omega = [\delta, \Omega],$$
$$\mathrm{d}\Omega(D_\mu, D_\nu, D_\xi) = D_\mu \Delta_{\nu,\xi} + D_\nu \Delta_{\xi,\mu} + D_\xi \Delta_{\mu,\nu} = 0$$

because $[\delta, \Omega](D_\mu, D_\nu, D_\xi) = 0$.

7. Some examples

Every principal bundle can be seen as an extension of the path bundle [6]. The bundle morphism is expressed

$$f(\pi_o^x) = \Pi(\pi G_o^x),$$

where $\Pi(\pi G_o^x)$ is the final point of the horizontal path in GM, that is projected over Π_o^x. We also have the following identity between the connections:

$$f^*\theta = \mathrm{d}H \circ \delta.$$

Helped with this formula, the identities presented in [8] are,

$$\delta_\mu(x)H(\gamma) = A_\mu(x)H(\gamma) = \mathrm{d}H \circ \delta(\tilde{D}_\mu);$$
$$\Delta_{\mu,\nu}(\pi_o^x)H(\gamma) = F_{\mu\nu}(x)H(\gamma) = \mathrm{d}H \circ \Omega(D_\mu, D_\nu).$$

As another example we consider the vector fields in $\mathcal{L}$, $C(X)$, where X is a vector field in M defined as

$$C(X)f(\gamma) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} f(\varphi_X(s) \circ \gamma),$$

we prove that

$$[C(X), C(Y)] = C([X, Y])$$

that is, the operators $C(X)$ satisfy the commutation relations of the diffeomorphism constraint of general relativity.

The group of diffeomorphisms of M, $\mathrm{Diff}(M)$, has a structure of differentiable space and given a vector field X in M we can associate a vector field $V(X)$ in $\mathrm{Diff}(M)$. Let ϕ: $\mathrm{Diff}(M) \times \mathcal{L} \to \mathcal{L}$ be the action of $\mathrm{Diff}(M)$ in $\mathcal{L}$ and let $\bar{V}(X)$ be the vector field $(V(X), 0)$ in $\mathrm{Diff}(M) \times \mathcal{L}$, then it follows that $\mathrm{d}\phi \circ \bar{V}(X) = C(X) \circ \phi$; the proof follows by showing that $[V(X), V(Y)] = V([X, Y])$ and the familiar identity $[\mathrm{d}\phi \circ Z_1, \mathrm{d}\phi \circ Z_2] = [Z_1, Z_2] \circ \phi$.

8. Discussion

We end with a discussion section. To begin with we note the main points of the paper. We have tried to show the utility of differentiable spaces and the calculus therein developed to rigorous and unified proofs of well established results and tools. We detach it in a clear definition and local triviality of the tangent bundle of the group of loops and the path bundle given by the respective definitions of their plots, a well established and local formula for differential forms and the exterior derivative, and a unified viewpoint for all the derivatives in particular proving that the loop derivative is the curvature of the *universal connection 1-form*. Secondly, we want to mention these tools in the context of the loop representation of quantum gravity. It is well known that the Einstein equations being diffeomorphism invariant imposes serious constraints on a possible theory of quantum gravity. For instance, the loop representation of it must be written in terms of knots invariants (allowing intersections on them). This seems to be at odds with our point of view because we are using a differentiable perspective but it seems that quantum gravity being expressed as functions of the topology class of a loop can never be differentiable at all. In spite of this appearance new results [7] seem to show that on extending loop functions to distributions, Vassiliev invariants are 'loop differentiable'. These results would serve (tentatively) as a differentiable arena for quantum gravity. Although our concept of differentiable function is not distributional, it seems that much of the work in this paper could be done along these lines. The differentiable way of thinking of quantum gravity is still promising.

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